Systems of conservation laws within the framework of the projective theory of congruences: the Lévy transformations of semi-Hamiltonian systems

2000 J. Phys. A: Math. Gen. 33 6935

(http://iopscience.iop.org/0305-4470/33/39/309)

J. Phys. A: Math. Gen. **33** (2000) 6935–6952. Printed in the UK PII: S0305-4470(00)14451-8

Systems of conservation laws within the framework of the projective theory of congruences: the Lévy transformations of semi-Hamiltonian systems

E V Ferapontov

Department of Mathematical Sciences, Loughborough University, Loughborough, Leicestershire LE11 3TU, UK
and
Centre for Nonlinear Studies, Landau Institute of Theoretical Physics, Academy of Science of Russia, Kosygina 2, 117940 Moscow, GSP-1, Russia

E-mail: E.V.Ferapontov@lboro.ac.uk

Received 1 June 2000, in final form 17 July 2000

Abstract. We continue the investigation of the recently proposed geometric correspondence between systems of conservation laws and congruences of lines in projective space. The relationship between the 'additional' conservation laws and hypersurfaces conjugate to a congruence is established, thus providing a basis for the Lévy transformations of semi-Hamiltonian systems. Similarly, the correspondence between commuting flows and certain families of planes (containing the lines of the congruence) gives rise to the adjoint Lévy transformations.

1. Introduction

It was observed recently that many constructions of the theory of hyperbolic systems of conservation laws

$$u^i_t = f^i(u)_x \qquad i = 1, \ldots, n \tag{1}$$

are, in a sense, parallel to that of the projective theory of congruences. The correspondence proposed in [1, 2] associates with any system (1) a congruence of lines

$$y^i = u^i y^0 - f^i(u) \qquad i = 1, \ldots, n \tag{2}$$

in an $(n+1)$-dimensional projective space with coordinates $y^0, \ldots, y^n$. It turns out that the basic concepts of the theory of systems of conservation laws, such as the shock and rarefaction curves, Riemann invariants, reciprocal transformations and systems of Temple class [15] acquire a clear and simple projective interpretation when reformulated in the language of the theory of congruences. For instance, this correspondence enabled the classification of systems of Temple class to be reduced to a much simpler geometric problem of the classification of congruences with either planar or conical developable surfaces. In particular, the results of [15] became intuitive geometric statements about families of lines in projective space. Another application of the correspondence proposed was the construction of the Laplace transformations of hydrodynamic-type systems in Riemann invariants [7] which, on the geometric level, have been the subject of extensive research in projective differential geometry.

In this paper we continue the investigation of the correspondence between systems of conservation laws and congruences of lines in projective space and introduce the Lévy and adjoint Lévy transformations of hydrodynamic-type systems in Riemann invariants. Along with Laplace transformations, they satisfy a number of remarkable geometric and algebraic properties.

In sections 2–4 we recall the necessary information about conservation laws, commuting flows and Riemann invariants of systems (1). Section 5 provides a simple projective interpretation of the 'additional' conservation laws

$$h(u)_t = g(u)_x. \tag{3}$$

Namely, with any conservation law (3) we associate a hypersurface with the parametric equations

$$y^0 = \frac{g(u)}{h(u)} \qquad y^i = u^i \frac{g(u)}{h(u)} - f^i(u) \qquad i = 1, \ldots, n.$$

Based on the results of section 1 we demonstrate that this hypersurface is conjugate to a congruence (2), and any such hypersurface can be obtained within this construction.

A similar geometric correspondence between commuting flows of system (1) and certain n-parameter families of planes containing the lines of congruence (2) is discussed in section 6. In the two-component case ($n = 2$) this construction provides an explicit parametrization of surfaces harmonic to a congruence (2) by commuting flows of system (1).

The results of sections 5 and 6 allow us to introduce, in a purely geometric way, Lévy and adjoint Lévy transformations of hydrodynamic-type systems in Riemann invariants,

$$R^i_t = \lambda^i(R) R^i_x \qquad i = 1, \ldots, n \tag{4}$$

whose characteristic velocities $\lambda^i(R)$ satisfy the semi-Hamiltonian property

$$\partial_k \left(\frac{\partial_j \lambda^i}{\lambda^j - \lambda^i} \right) = \partial_j \left(\frac{\partial_k \lambda^i}{\lambda^k - \lambda^i} \right) \qquad i \neq j \neq k \quad \partial_i = \partial/\partial R^i \tag{5}$$

(according to the results of [14] this implies the integrability of system (4)). Let us choose any conservation law

$$h(R)_t = g(R)_x$$

of system (4) and introduce the new system

$$R^i_t = \Lambda^i(R) R^i_x \qquad i = 1, \ldots, n \tag{6}$$

with the characteristic velocities $\Lambda^i(R)$ defined by the formulae

$$\Lambda^\alpha = \frac{g}{h} \qquad \Lambda^i = \frac{\lambda^i \partial_\alpha h - a_{i\alpha} g}{\partial_\alpha h - a_{i\alpha} h} \qquad i \neq \alpha \tag{7}$$

where $a_{i\alpha} = \frac{\partial_\alpha \lambda^i}{\lambda^\alpha - \lambda^i}$ (here the index α plays a distinguished role). The system (6) and (7) is called the Lévy transform $\mathcal{L}_\alpha$ of system (4).

Similarly, let us take a commuting flow

$$R^i_t = \mu^i(R) R^i_x \qquad i = 1, \ldots, n$$

of system (4) and introduce the new system (6) with the characteristic velocities $\Lambda^i(R)$ defined by the formulae

$$\Lambda^\alpha = \frac{\lambda^\alpha \partial_\alpha \mu^\alpha - \mu^\alpha \partial_\alpha \lambda^\alpha}{\partial_\alpha \mu^\alpha} \qquad \Lambda^i = \frac{\lambda^i \mu^\alpha - \lambda^\alpha \mu^i}{\mu^\alpha - \mu^i} \qquad i \neq \alpha \tag{8}$$

which again depend on the choice of α. The system (6) and (8) is called the adjoint Lévy transform $\mathcal{L}^*_\alpha$ of system (4). Transformations $\mathcal{L}_\alpha$ and $\mathcal{L}^*_\alpha$ satisfy a number of remarkable properties which are briefly reviewed in sections 7 and 8. In particular, both of them preserve the semi-Hamiltonian condition (5). In [4] the transformations $\mathcal{L}_\alpha$ and $\mathcal{L}^*_\alpha$ have been identified with the vertex operators of a multicomponent KP hierarchy.

A closely related construction of Ribaucour congruences of spheres is discussed in the appendix.

2. Systems of conservation laws. Equations for the conserved densities

We consider hyperbolic systems of conservation laws

$$u^i_t = f^i(u)_x = v^i_j(u)u^j_x \qquad v^i_j = \frac{\partial f^i}{\partial u^j} \tag{9}$$

assuming the eigenvalues λ^i of the matrix v^i_j (called the characteristic velocities of system (9)) to be real and pairwise distinct. Let $\vec{\xi}_i = (\xi^1_i(u), \ldots, \xi^n_i(u))^t$ be the corresponding eigenvectors:

$$v\vec{\xi}_i = \lambda^i \vec{\xi}_i \quad \text{or, in components,} \quad v^s_k \xi^k_i = \lambda^i \xi^s_i.$$

We denote by $L_i = \xi^k_i \frac{\partial}{\partial u^k}$ the Lie derivative along the vector field $\vec{\xi}_i$ and introduce the commutator expansions

$$[L_i, L_j] = L_i L_j - L_j L_i = c^k_{ij} L_k$$

where c^k_{ij} are certain functions of u. Let

$$h(u)_t = g(u)_x$$

be any conservation law of system (9). Its density h and flux g satisfy the equations

$$\frac{\partial g}{\partial u^k} = \frac{\partial h}{\partial u^s} v^s_k$$

which upon contraction with $\vec{\xi}_i = (\xi^1_i, \ldots, \xi^n_i)^t$ result in

$$\frac{\partial g}{\partial u^k} \xi^k_i = \frac{\partial h}{\partial u^s} v^s_k \xi^k_i$$

or

$$L_i g = \lambda^i L_i h \qquad i = 1, \ldots, n. \tag{10}$$

Equations (10) are the defining equations for the conserved densities h and the corresponding fluxes g. The compatibility conditions of (10) are of the form

$$L_i(L_j g) - L_j(L_i g) = c^k_{ij} L_k g$$

or, taking into account (10),

$$L_i(\lambda^j L_j h) - L_j(\lambda^i L_i h) = c^k_{ij} \lambda^k L_k h.$$

This results in the following linear second-order system for the conserved densities h:

$$L_i L_j h = \frac{L_j \lambda^i}{\lambda^j - \lambda^i} L_i h + \frac{L_i \lambda^j}{\lambda^i - \lambda^j} L_j h + c^k_{ij} \frac{\lambda^i - \lambda^k}{\lambda^i - \lambda^j} L_k h \qquad i \neq j. \tag{11}$$

In particular, $h = u^1, \dots, u^n$ satisfy (11). It should be pointed out that in the generic situation (to be more precise, in the case $c^k_{ij} \neq 0$ for any $i \neq j \neq k$) the space of solutions of the overdetermined linear system (11) is finite dimensional. In what follows we will make use of the equations satisfied by the ratio $\varphi = \frac{g}{h}$, which can be obtained by rewriting (10) in the form

$$L_i(\varphi h) = \lambda^i L_i h$$

or, equivalently,

$$L_i \ln h = \frac{L_i \varphi}{\lambda^i - \varphi}. \tag{12}$$

The compatibility conditions of (12) imply the following nonlinear second-order system for φ:

$$L_i L_j \varphi = \left(\frac{1}{\varphi - \lambda^i} + \frac{1}{\varphi - \lambda^j} \right) L_i \varphi L_j \varphi + \frac{L_j \lambda^i}{\lambda^j - \lambda^i} \frac{\varphi - \lambda^j}{\varphi - \lambda^i} L_i \varphi + \frac{L_i \lambda^j}{\lambda^i - \lambda^j} \frac{\varphi - \lambda^i}{\varphi - \lambda^j} L_j \varphi$$
$$+ c^k_{ij} \frac{\lambda^i - \lambda^k}{\lambda^i - \lambda^j} \frac{\varphi - \lambda^j}{\varphi - \lambda^k} L_k \varphi. \tag{13}$$

Formula (12) establishes an equivalence between the linear system (11) and the nonlinear system (13). The ratio $\varphi = \frac{g}{h}$ naturally arises in projective differential geometry (describing surfaces conjugate to a congruence, see section 3), and in the Lie sphere geometry (parametrizing Ribaucour congruences of spheres, see the appendix).

3. Commuting flows

A system of conservation laws

$$u^i_\tau = q^i(u)_x = w^i_j(u) u^j_x \qquad w^i_j = \frac{\partial q^i}{\partial u^j} \tag{14}$$

is called the commuting flow of system (9) if $u^i_{t\tau} = u^i_{\tau t}$ or, equivalently,

$$\left(\frac{\partial f^i}{\partial u^j} \frac{\partial q^j}{\partial u^k} u^k_x \right)_x = \left(\frac{\partial q^i}{\partial u^j} \frac{\partial f^j}{\partial u^k} u^k_x \right)_x.$$

Equating the coefficients at u^k_{xx}, we arrive at the commutativity of the matrices $v = v^i_j$ and $w = w^i_j$. Thus, they have the same eigenvectors $\vec{\xi}_i$. Let μ^i be the characteristic velocities of system (14):

$$w \vec{\xi}_i = \mu^i \vec{\xi}_i.$$

According to section 2, the conserved densities h of system (14) satisfy the equations

$$L_i L_j h = \frac{L_j \mu^i}{\mu^j - \mu^i} L_i h + \frac{L_i \mu^j}{\mu^i - \mu^j} L_j h + c^k_{ij} \frac{\mu^i - \mu^k}{\mu^i - \mu^j} L_k h. \tag{15}$$

Since both systems (11) and (15) share a common set of n functionally independent solutions $h = u^1, \dots, u^n$, their coefficients must coincide identically (if this were not the case, there would be a first-order relation between $L_i h$, contradicting the functional independence of $u^1, \dots, u^n$). Thus,

$$\frac{L_j \mu^i}{\mu^j - \mu^i} = \frac{L_j \lambda^i}{\lambda^j - \lambda^i} \qquad \text{for any} \quad i \neq j \tag{16}$$

and

$$c_{ij}^k\left(\frac{\mu^i-\mu^k}{\mu^i-\mu^j}-\frac{\lambda^i-\lambda^k}{\lambda^i-\lambda^j}\right)=0 \qquad \text{for any} \quad i\neq j\neq k. \tag{17}$$

In this form the equations governing commuting flows of system (9) have been set down in [13].

If $n = 2$ equations (17) are redundant. Let us consider the case $n = 3$ and assume that at least one of the coefficients c_{ij}^k (with three distinct indices i, j, k) is non-zero. Then equations (17) imply $\mu^i = \lambda^i b - a$ for appropriate b and a. The substitution of this representation into (16) implies, however, that both a and b must be constants so that the commuting flow is trivial. Hence, for $n = 3$ only systems with zero c_{ij}^k (for distinct i, j, k) may possess non-trivial commuting flows.

Similarly, in the case $n \geqslant 3$ the presence of 'sufficiently many' non-zero coefficients c_{ij}^k prevents the existence of non-trivial commuting flows.

4. Diagonalizable systems of conservation laws

Let us assume that all coefficients c_{ij}^k (with distinct i, j, k) are zero. In this case one can normalize the eigenvectors $\vec{\xi}_i$ in such a way that the Lie derivatives L_i will pairwise commute: $[L_i, L_j] = 0$, so that the remaining coefficients c_{ij}^j will also be zero. The commutativity of L_i implies the existence of the coordinates $R^1(u), \ldots, R^n(u)$ such that L_i become partial derivatives: $L_i = \partial_i = \partial/\partial R^i$. In these coordinates equations (9) assume the diagonal form

$$R_t^i = \lambda^i(R)R_x^i \qquad i = 1, \ldots, n. \tag{18}$$

Variables R^i are called the Riemann invariants of system (9). Systems (9), possessing Riemann invariants, are called diagonalizable. Let $u_t = f_x$ be a conservation law of system (18). In the diagonalizable case equations (10) assume the form

$$\partial_i f = \lambda^i \partial_i u \qquad i = 1, \ldots, n$$

while system (11) for the conserved densities u simplifies to

$$\partial_i\partial_j u = a_{ij}\partial_i u + a_{ji}\partial_j u \qquad i \neq j \tag{19}$$

where $a_{ij} = \frac{\partial_j\lambda^i}{\lambda^j-\lambda^i}$. The compatibility conditions of system (19) are of the form

$$\partial_k a_{ij} = a_{ik}a_{kj} + a_{ij}a_{jk} - a_{ij}a_{ik} \qquad i \neq j \neq k \tag{20}$$

they must be identically satisfied if we require system (19) to possess n functionally independent solutions $u = u^1, \ldots, u^n$. In fact, conditions (20) imply the existence of infinitely many conservation laws parametrized by n arbitrary functions of one variable. Systems (18) satisfying (20) are called semi-Hamiltonian. We refer to [5, 13, 14] for further information concerning integrability, differential geometry and applications of the semi-Hamiltonian systems. Semi-Hamiltonian systems possess infinitely many commuting flows

$$R_\tau^i = \mu^i R_x^i$$

with the characteristic velocities μ^i governed by the equations

$$\frac{\partial_j\mu^i}{\mu^j-\mu^i} = \frac{\partial_j\lambda^i}{\lambda^j-\lambda^i} = a_{ij} \qquad i \neq j.$$

We point out that any semi-Hamiltonian system possesses infinitely many different conservative representations.

5. Systems of conservation laws and congruences of lines. Hypersurfaces conjugate to a congruence

With any system of conservation laws

$$u^i_t = f^i(u)_x$$

we associate an n-parameter family of lines

$$\begin{aligned} y^1 &= u^1 y^0 - f^1(u) \\ &\vdots \\ y^n &= u^n y^0 - f^n(u) \end{aligned} \tag{21}$$

in the $(n+1)$-dimensional space A^{n+1} with the coordinates $y^0, y^1, \ldots, y^n$ (see [1, 2] for the motivation and the most important properties of this correspondence). In the case $n = 2$ we obtain a two-parameter family of lines, or a congruence of lines in A^3. From the beginning of the 19th century the theory of congruences was one of the most popular chapters of classical differential geometry (see, e.g., [9]). We keep the name 'congruence' for any n-parameter family of lines (21). Let us briefly recall the main geometrical properties of line congruences.

Definition. A hypersurface $M^n \subset A^{n+1}$ is said to be focal to the congruence (21) if all lines of the congruence are tangent to M^n.

The idea of focal hypersurfaces is obviously borrowed from optics: thinking of the lines of the congruence as the rays of light, one can intuitively think of the focal hypersurfaces as the locus in A^{n+1} where the light concentrates (this explains why in German literature focal hypersurfaces are called 'Brennflächen', which can be translated as 'burning surfaces'). It can be demonstrated that the generic congruence (21) in A^{n+1} possesses exactly n focal hypersurfaces, so that any congruence can be viewed as a collection of common tangents to n hypersurfaces in A^{n+1}. The radius-vector $\vec{r}_i$ of the ith focal hypersurface is given by parametric equations

$$\vec{r}_i = (y^0, y^1, \ldots, y^n) = (\lambda^i, u^1\lambda^i - f^1, \ldots, u^n\lambda^i - f^n) \tag{22}$$

where λ^i is the ith characteristic velocity of system (9), see [1, 2]. A line (21) is tangent to $\vec{r}_i$ at the point with $y^0 = \lambda^i$. Indeed, substituting $y^0 = \lambda^i$ in (21) we obviously obtain a point belonging to the focal hypersurface $\vec{r}_i$. Moreover, the formula

$$L_i \vec{r}_i = L_i \lambda^i (1, u^1, \ldots, u^n)$$

(which is a consequence of (10)) implies that the direction $L_i \vec{r}_i$ coincides with the direction of the line (21), thus guaranteeing the tangency. Summarizing, we have an explicit parametrization (22) of the focal hypersurfaces $\vec{r}_i$ by the characteristic velocities λ^i of system (9).

Let us consider a hypersurface M^n with the radius-vector $\vec{r}$ parametrized as follows:

$$\vec{r} = (y^0, y^1, \ldots, y^n) = \left(\varphi, u^1\varphi - f^1, \ldots, u^n\varphi - f^n\right) \tag{23}$$

here $\varphi(u)$ is an arbitrary function which is assumed to be different from λ^i so that M^n is not focal. A line (21) meets M^n at the point with $y^0 = \varphi$. Obviously, any hypersurface $M^n \in A^{n+1}$ can be parametrized in the form (23) for an appropriate function φ. We say that the hypersurface M^n is conjugate to the congruence (21) if and only if

$$L_i L_j \vec{r} \in TM^n \qquad \text{for any } i \neq j.$$

Geometrically, this means that the developable surfaces of the congruence (21) meet M^n at the curves of a conjugate net. In a 3-space the notion of the conjugacy between a surface and a congruence was introduced by Guichard (see [6], chapter 1; [9]).

Theorem 1. *Hypersurface (23) is conjugate to a congruence if and only if φ is representable in the form $\varphi = \frac{g}{h}$, where $h_t = g_x$ is a conservation law of system (9).*

Proof. The tangent space of M^n is spanned by the vectors

$$L_j\vec{r} = (L_j\varphi)\vec{U} + (\varphi - \lambda^j)L_j\vec{U} \tag{24}$$

where $\vec{U}$ denotes the $(n+1)$-vector $(1, u^1, \ldots, u^n)$. Hence,

$$L_j\vec{U} = \frac{L_j\varphi}{\lambda^j - \varphi}\vec{U} \quad \mathrm{mod}\, TM^n. \tag{25}$$

Let us compute $L_iL_j\vec{r}$:

$$L_iL_j\vec{r} = (L_iL_j\varphi)\vec{U} + (L_j\varphi)L_i\vec{U} + L_i(\varphi - \lambda^j)L_j\vec{U} + (\varphi - \lambda^j)L_iL_j\vec{U}.$$

Inserting here $L_iL_j\vec{U}$ from (11) and keeping in mind (25), we arrive at

$$L_iL_j\vec{r} = \left(L_iL_j\varphi + L_j\varphi\frac{L_i\varphi}{\lambda^i - \varphi} + L_i(\varphi - \lambda^j)\frac{L_j\varphi}{\lambda^j - \varphi} + (\varphi - \lambda^j)\left(\frac{L_j\lambda^i}{\lambda^j - \lambda^i}\frac{L_i\varphi}{\lambda^i - \varphi}\right.\right.$$
$$\left.\left. + \frac{L_i\lambda^j}{\lambda^i - \lambda^j}\frac{L_j\varphi}{\lambda^j - \varphi} + c^k_{ij}\frac{\lambda^i - \lambda^k}{\lambda^i - \lambda^j}\frac{L_k\varphi}{\lambda^k - \varphi}\right)\right)\vec{U} \quad \mathrm{mod}\, TM^n.$$

Hence, $L_iL_j\vec{r} \in TM^n$ if and only if the coefficient at $\vec{U}$ vanishes. The resulting system for φ coincides identically with (13).

Thus, hypersurfaces conjugate to a congruence (21) are parametrized by conservation laws of system (9). According to [6], two hypersurfaces conjugate to one and the same congruence are said to be in the relation F (or related by a fundamental transformation). □

Remark 1. The case $\varphi = \lambda^i$ requires a special treatment. In this case M^n coincides with the ith focal hypersurface of a congruence. A direct computation shows that the ith focal hypersurface is conjugate to a congruence if and only if $c^i_{jk} = 0$ for any $j, k \neq i$ (i is fixed!). This is equivalent to the existence of a function $R^i(u)$ (called the ith Riemann invariant) such that

$$R^i_t = \lambda^i R^i_x$$

in particular, all focal hypersurfaces are conjugate to a congruence if and only if the system (9) possesses n Riemann invariants. The proof and some further details can be found in [2], see also [3].

Remark 2. If the conservation law $h_t = g_x$ is a linear combination of conservation laws (9), the hypersurface M^n degenerates into a hyperplane (which is automatically conjugate to any congruence). Thus, only 'additional' conservation laws give rise to the non-trivial conjugate hypersurfaces.

Remark 3. Conjugate hypersurfaces always appear in one-parameter families since, for a fixed density h, one can add a constant c to the flux g. The corresponding family of conjugate hypersurfaces $\vec{r}_c$ determined by $\varphi_c = \frac{g+c}{h}$ forms a parallel family, that is, the directions $L_j\vec{r}_c$ are independent of c. This immediately follows from (24) since the ratio $\frac{L_j\varphi_c}{\varphi_c - \lambda^j} = -\frac{L_jh}{h}$ does not depend on c.

6. Surfaces harmonic to a congruence

In this section we consider two-component systems of conservation laws

$$u^1_t = f^1_x \qquad u^2_t = f^2_x \tag{26}$$

and the associated congruences of lines in A^3:

$$y^1 = u^1 y^0 - f^1 \qquad y^2 = u^2 y^0 - f^2. \tag{27}$$

Let

$$u^1_\tau = q^1_x \qquad u^2_\tau = q^2_x \tag{28}$$

be a commuting flow of system (26). In the Riemann invariants R^1, R^2 (we point out that any two-component system is diagonalizable) equations (26) and (28) assume the forms

$$R^1_t = \lambda^1 R^1_x \qquad R^2_t = \lambda^2 R^2_x$$

and

$$R^1_\tau = \mu^1 R^1_x \qquad R^2_\tau = \mu^2 R^2_x$$

respectively. Here the densities $u = (u^1, u^2)$ and the fluxes $f = (f^1, f^2)$, $q = (q^1, q^2)$ satisfy the equations

$$\partial_i f = \lambda^i \partial_i u \qquad \partial_i q = \mu^i \partial_i u \qquad i = 1, 2.$$

With any commuting flow (28) we associate a two-parameter family of planes in A^3 defined by the equations

$$\frac{y^1 - u^1 y^0 + f^1}{q^1} = \frac{y^2 - u^2 y^0 + f^2}{q^2}. \tag{29}$$

The family of planes (29) satisfies the following remarkable properties.

(a) Each plane π from the family (29) contains a line l of the congruence (27).
(b) The congruence of lines $l_1 = \pi \cap \partial_1 \pi$ is conjugate to the focal surface $\vec{r}_1$ of the congruence (27). Similarly, the congruence of lines $l_2 = \pi \cap \partial_2 \pi$ is conjugate to $\vec{r}_2$. The lines l_1 and l_2 are called the characteristics of the plane π. The characteristic l_1 (respectively, l_2), meets the line l in the point of tangency of l with the focal surface $\vec{r}_1$ (respectively, $\vec{r}_2$).

The proof follows from the explicit parametrization of the congruences l_1, l_2.
Congruence l_1:

$$y^1 = \left(u^1 - \frac{q^1}{\mu^1}\right) y^0 - \left(f^1 - \frac{\lambda^1 q^1}{\mu^1}\right)$$
$$y^2 = \left(u^2 - \frac{q^2}{\mu^1}\right) y^0 - \left(f^2 - \frac{\lambda^1 q^2}{\mu^1}\right).$$

Congruence l_2:

$$y^1 = \left(u^1 - \frac{q^1}{\mu^2}\right) y^0 - \left(f^1 - \frac{\lambda^2 q^1}{\mu^2}\right)$$

$$y^2 = \left(u^2 - \frac{q^2}{\mu^2}\right) y^0 - \left(f^2 - \frac{\lambda^2 q^2}{\mu^2}\right).$$

Obviously, the line l_1 passes through the point

$$(y^0, y^1, y^2) = \left(\lambda^1, u^1\lambda^1 - f^1,\ u^2\lambda^1 - f^2\right)$$

of the focal surface $\vec{r}_1$. Similarly, the line l_2 passes through the point

$$(y^0, y^1, y^2) = \left(\lambda^2, u^1\lambda^2 - f^1,\ u^2\lambda^2 - f^2\right)$$

of the focal surface $\vec{r}_2$. The point of intersection $l_1 \cap l_2 \in \pi$ has the coordinates

$$\begin{aligned} y^0 &= \frac{\lambda^2\mu^1 - \lambda^1\mu^2}{\mu^1 - \mu^2} \\ y^1 &= \frac{\lambda^2\mu^1 - \lambda^1\mu^2}{\mu^1 - \mu^2} u^1 + \frac{\lambda^1 - \lambda^2}{\mu^1 - \mu^2} q^1 - f^1 \\ y^2 &= \frac{\lambda^2\mu^1 - \lambda^1\mu^2}{\mu^1 - \mu^2} u^2 + \frac{\lambda^1 - \lambda^2}{\mu^1 - \mu^2} q^2 - f^2 \end{aligned} \tag{30}$$

and sweeps a surface in A^3. By a construction, the surface (30) is the envelope of the family of planes (29). It has the following geometric properties.

(a) Each tangent plane π of the surface (30) contains a line l of the congruence (27). By construction, π and $l \in \pi$ correspond to the same values of parameters R^1, R^2. Thus, one can speak of the correspondence between lines (27) and points of the surface (30).
(b) The net R^1, R^2 on the surface (30) is conjugate. In other words, the developable surfaces of the congruence (27) correspond to a conjugate net on the surface (30).

Surfaces satisfying the properties (a) and (b) are called harmonic to a congruence (27), see [9], p 251. Formulae (30) provide an explicit parametrization of such surfaces by the commuting flows of system (26). Conversely, any surface harmonic to a congruence (27) is representable in the form (30).

7. Lévy transformations of semi-Hamiltonian systems

Let us consider a semi-Hamiltonian system (18) in Riemann invariants

$$R^i_t = \lambda^i(R) R^i_x \qquad i = 1, \ldots, n$$

whose conservation laws

$$u_t = f_x$$

satisfy the equations

$$\partial_i f = \lambda^i \partial_i u \qquad i = 1, \ldots, n$$

$$\partial_i \partial_j u = a_{ij} \partial_i u + a_{ji} \partial_j u \qquad i \neq j \qquad a_{ij} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}.$$

Let us choose a particular conservation law

$$h_t = g_x$$

of system (18) and introduce the new variable U by the formula

$$U = u - \frac{h}{\partial_\alpha h}\partial_\alpha u \tag{31}$$

where α is fixed. Transformations of this type originating from a projective differential geometry of conjugate nets are known as Lévy transformations [6, chapter 1, 10, p 94, 12]. In [4] Lévy transformations have been identified with the vertex operators of multicomponent KP hierarchy. Their geometric interpretation will be clarified in the second half of this section. We will refer to (31) as Lévy transformation $\mathcal{L}_\alpha$. A direct calculation shows that $U = \mathcal{L}_\alpha(u)$ satisfies the equations of the same form as u:

$$\partial_i\partial_j U = A_{ij}\partial_i U + A_{ji}\partial_j U \tag{32}$$

where the new coefficients $A = \mathcal{L}_\alpha(a)$ are given by the formulae

$$A_{\alpha i} = \left(1 - \frac{a_{i\alpha}h}{\partial_\alpha h}\right)\frac{\partial_i h}{h} \qquad i \neq \alpha$$

$$A_{ij} = a_{ij} + \partial_j \ln\left(1 - \frac{a_{i\alpha}h}{\partial_\alpha h}\right) \qquad i \neq \alpha \quad j \text{ is arbitrary.}$$

Transformations $\mathcal{L}_\alpha$ can be pulled back to the transformations of the corresponding hydrodynamic-type systems: let us introduce the system

$$R^i_T = \Lambda^i(R)R^i_X \qquad i = 1, \ldots, n \tag{33}$$

with the characteristic velocities

$$\begin{aligned} \Lambda^\alpha &= \frac{g}{h} \\ \Lambda^i &= \frac{\lambda^i\partial_\alpha h - a_{i\alpha}g}{\partial_\alpha h - a_{i\alpha}h} \qquad i \neq \alpha. \end{aligned} \tag{34}$$

Theorem 2. *Conservation laws*

$$U_T = F_X$$

of the system (33) and (34) are the $\mathcal{L}_\alpha$-transforms of conservation laws

$$u_t = f_x$$

of system (18):

$$U = \mathcal{L}_\alpha(u) = u - \frac{h}{\partial_\alpha h}\partial_\alpha u$$

$$F = \mathcal{L}_\alpha(f) = f - \frac{g}{\partial_\alpha g}\partial_\alpha f.$$

Formally, the proof of this theorem follows from the identities

$$\partial_i F = \Lambda^i \partial_i U \qquad A_{ij} = \frac{\partial_j \Lambda^i}{\Lambda^j - \Lambda^i}$$

which can be verified by a direct calculation. Geometric constructions underlying these formulae will be discussed below. The system (33) and (34) will be called the $\mathcal{L}_\alpha$-transform of system (18). Obviously, transformations $\mathcal{L}_\alpha$ preserve the semi-Hamiltonian property.

We also include the Lévy transforms of Lame coefficients h_i defined by the formulae

$$\partial_j \ln h_i = a_{ij} \qquad j \neq i.$$

The $\mathcal{L}_\alpha$-transformed Lame coefficients are given by

$$H_\alpha = h_\alpha \frac{h}{\partial_\alpha h}$$

$$H_i = h_i \left(1 - \frac{a_{i\alpha} h}{\partial_\alpha h}\right) \qquad i \neq \alpha.$$

One can check directly that

$$\partial_j \ln H_i = A_{ij} \qquad j \neq i.$$

Lévy transformations of hydrodynamic-type systems in Riemann invariants are closely related to Laplace transformations discussed recently in [7, 11]. We recall that the Laplace transformation $S_{\alpha\beta}$ of system (19) is defined by the formula

$$U = S_{\alpha\beta}(u) = u - \frac{\partial_\alpha u}{a_{\beta\alpha}}$$

where both indices $\alpha \neq \beta$ are fixed. Laplace transformations also induce transformations of the characteristic velocities λ^i, the explicit form of which has been set down in [7]. One can check directly that the Lévy transformation $\mathcal{L}_\alpha$ of system (18) is related to its Lévy transformation $\mathcal{L}_\beta$ via the Laplace transformation $S_{\alpha\beta}$:

$$\mathcal{L}_\alpha = S_{\alpha\beta} \circ \mathcal{L}_\beta.$$

To clarify the geometric picture underlying transformations $\mathcal{L}_\alpha$ we choose an arbitrary conservative representation

$$u^i_t = f^i_x$$

of system (18) and introduce the associated congruence

$$y^1 = u^1 y^0 - f^1$$

$$\vdots$$

$$y^n = u^n y^0 - f^n.$$

Let M^n be a hypersurface conjugate to this congruence. Following section 4, we represent the radius-vector $\vec{r}$ of M^n in the form

$$\vec{r} = \left(\varphi, u^1\varphi - f^1, \ldots, u^n\varphi - f^n\right) \qquad \varphi = \frac{g}{h}$$

where $h_t = g_x$ is a conservation law of system (18). The coordinate system $R^1, \ldots, R^n$ on M^n is conjugate, so that

$$\partial_i \partial_j \vec{r} \in TM^n \qquad \text{for any} \quad i \neq j.$$

Let us introduce a new congruence consisting of the tangents to the R^α-curves on the hypersurface M^n. Parametrically, its lines can be represented in the form

$$\vec{r} + t\partial_\alpha \vec{r}$$

or, in the components,

$$
\begin{aligned}
y^0 &= \varphi + t\partial_\alpha\varphi \\
y^1 &= u^1\varphi - f^1 + t(u^1\partial_\alpha\varphi + (\varphi - \lambda^\alpha)\partial_\alpha u^1) \\
&\vdots \\
y^n &= u^n\varphi - f^n + t(u^n\partial_\alpha\varphi + (\varphi - \lambda^\alpha)\partial_\alpha u^n).
\end{aligned}
$$

Inserting $t = \frac{y^0-\varphi}{\partial_\alpha\varphi}$ in the last n equations, we arrive at the new congruence

$$
\begin{aligned}
y^1 &= U^1 y^0 - F^1 \\
&\vdots \\
y^n &= U^n y^0 - F^n
\end{aligned}
\tag{35}
$$

where

$$
\begin{aligned}
U^1 &= u^1 + \frac{\varphi - \lambda^\alpha}{\partial_\alpha\varphi}\partial_\alpha u^1 \qquad F^1 = f^1 + \varphi\frac{\varphi - \lambda^\alpha}{\partial_\alpha\varphi}\partial_\alpha u^1 \\
&\vdots \\
U^n &= u^n + \frac{\varphi - \lambda^\alpha}{\partial_\alpha\varphi}\partial_\alpha u^n \qquad F^n = f^n + \varphi\frac{\varphi - \lambda^\alpha}{\partial_\alpha\varphi}\partial_\alpha u^n.
\end{aligned}
$$

Since $\frac{\varphi-\lambda^\alpha}{\partial_\alpha\varphi} = -\frac{h}{\partial_\alpha h}$, these formulae can be rewritten in the form

$$
U = u - \frac{h}{\partial_\alpha h}\partial_\alpha u \qquad F = f - \frac{g}{\partial_\alpha g}\partial_\alpha f.
$$

Congruence (35) will be called the $\mathcal{L}_\alpha$-transform of the initial congruence. The corresponding system of conservation laws

$$
U^i_T = F^i_X
$$

has the same Riemann invariants $R^1, \ldots, R^n$:

$$
R^i_T = \Lambda^i R^i_X
$$

where Λ^i can be computed by the formula $\Lambda^i = \partial_i F/\partial_i U$. A direct calculation results in the formulae (34). Note that the final expressions for Λ^i do not depend on the particular conservative representation $u^i_t = f^i_x$ of system (18). If, for M^n, we choose any of the focal hypersurfaces of the congruence (which are all conjugate to a congruence if the system possesses Riemann invariants), the above construction reproduces Laplace transformations.

Formula (32) shows that the density $u = h$ belongs to the kernel of the Lévy transformation $\mathcal{L}_\alpha$. Nevertheless, transformations $\mathcal{L}_\alpha$ can be explicitly inverted, as we will demonstrate in the next section.

Let us conclude with the formula for the composition of Lévy transformations

$$
\mathcal{L} = \mathcal{L}_n \circ \cdots \circ \mathcal{L}_2 \circ \mathcal{L}_1
$$

corresponding to n particular linearly independent conservation laws $h^i_t = g^i_x \quad i = 1, \ldots, n$ of system (18). The composition is understood as follows. Let $u_t = f_x$ be an arbitrary conservation law of system (18). First of all, we apply to $u_t = f_x$ the transformation $\mathcal{L}_1$, corresponding to the first conservation law $h^1_t = g^1_x$. Secondly, we apply to the result of the first step the transformation $\mathcal{L}_2$, corresponding to the $\mathcal{L}_1$-transform of the conservation law

$h^2_t = g^2_x$. Proceeding in this way, we obtain the $\mathcal{L}$-transformed density $U = \mathcal{L}(u)$ and the flux $F = \mathcal{L}(f)$ in the following compact form:

$$U = \frac{\det\begin{pmatrix} u & \partial_1 u & \dots & \partial_n u \\ h^1 & \partial_1 h^1 & \dots & \partial_n h^1 \\ & & & \\ h^n & \partial_1 h^n & \dots & \partial_n h^n \end{pmatrix}}{\det\begin{pmatrix} \partial_1 h^1 & \dots & \partial_n h^1 \\ & & \\ \partial_1 h^n & \dots & \partial_n h^n \end{pmatrix}} \qquad F = \frac{\det\begin{pmatrix} f & \partial_1 f & \dots & \partial_n f \\ g^1 & \partial_1 g^1 & \dots & \partial_n g^1 \\ & & & \\ g^n & \partial_1 g^n & \dots & \partial_n g^n \end{pmatrix}}{\det\begin{pmatrix} \partial_1 g^1 & \dots & \partial_n g^1 \\ & & \\ \partial_1 g^n & \dots & \partial_n g^n \end{pmatrix}}. \tag{36}$$

Geometrically, the composition $\mathcal{L}_n \circ \cdots \circ \mathcal{L}_2 \circ \mathcal{L}_1$ corresponds to the following construction (compare with [9], pp 255–66): choose an arbitrary conservative representation

$$u^i_t = f^i_x$$

of system (18) and introduce the corresponding congruence (21):

$$y^i = u^i y^0 - f^i.$$

Let M_i, $i = 1, \dots, n$, be n hypersurfaces conjugate to congruence (21). According to section 3, they are parametrized by n particular conservation laws $h^i_t = g^i_x$ of system (18). Let TM_i be the tangent hyperplanes of hypersurfaces M_i in the points of intersection with the line (21). The intersection

$$TM_1 \cap \cdots \cap TM_n$$

defines a new line

$$y^i = U^i y^0 - F^i$$

where the formulae for $U = U^i$ and $F = F^i$ coincide with (36).

8. Adjoint Lévy transformations

We again consider semi-Hamiltonian systems (18)

$$R^i_t = \lambda^i(R) R^i_x$$

with the conservation laws

$$u_t = f_x$$

satisfying the equations

$$\partial_i f = \lambda^i \partial_i u$$

$$\partial_i \partial_j u = a_{ij} \partial_i u + a_{ji} \partial_j u \qquad a_{ij} = \frac{\partial_j \lambda^i}{\lambda^j - \lambda^i}.$$

Let

$$R^i_\tau = \mu^i(R) R^i_x \tag{37}$$

be a commuting flow of system (18):

$$\frac{\partial_j \mu^i}{\mu^j - \mu^i} = a_{ij}.$$

Let q be the flux of the density u, corresponding to this commuting flow:

$$u_\tau = q_x.$$

The flux q and the density u satisfy the equations

$$\partial_i q = \mu^i \partial_i u.$$

Let us introduce the new variable U by the formula

$$U = u - \frac{q}{\mu^\alpha} \tag{38}$$

where α is fixed. We will refer to (38) as the adjoint Lévy transformation $\mathcal{L}^*_\alpha$. A direct calculation shows that $U = \mathcal{L}^*_\alpha(u)$ satisfies the equations of the same form as u:

$$\partial_i \partial_j U = A_{ij} \partial_i U + A_{ji} \partial_j U$$

where the new coefficients $A = \mathcal{L}^*_\alpha(a)$ are given by the formulae

$$A_{\alpha i} = a_{\alpha i} + \partial_i \ln \frac{\partial_\alpha \mu^\alpha}{\mu^\alpha} \qquad i \neq \alpha$$

$$A_{ij} = a_{ij} + \partial_j \ln \left(1 - \frac{\mu^i}{\mu^\alpha}\right) \qquad i \neq \alpha \quad j \text{ is arbitrary.}$$

Transformations $\mathcal{L}^*_\alpha$ can be pulled back to the transformations of the corresponding hydrodynamic-type systems: let us introduce the system

$$R^i_T = \Lambda^i(R) R^i_X \qquad i = 1, \ldots, n \tag{39}$$

with the characteristic velocities

$$\begin{aligned} \Lambda^\alpha &= \frac{\lambda^\alpha \partial_\alpha \mu^\alpha - \mu^\alpha \partial_\alpha \lambda^\alpha}{\partial_\alpha \mu^\alpha} \\ \Lambda^i &= \frac{\lambda^i \mu^\alpha - \lambda^\alpha \mu^i}{\mu^\alpha - \mu^i} \qquad i \neq \alpha. \end{aligned} \tag{40}$$

Theorem 3. *Conservation laws*

$$U_T = F_X$$

of the system (39) and (40) are the $\mathcal{L}^*_\alpha$-transforms of conservation laws

$$u_t = f_x$$

of system (18):

$$U = \mathcal{L}^*_\alpha(u) = u - \frac{q}{\mu^\alpha}$$

$$F = \mathcal{L}^*_\alpha(f) = f - \frac{\lambda^\alpha q}{\mu^\alpha}.$$

Formally, the proof of this theorem follows from the identities

$$\partial_i F = \Lambda^i \partial_i U \qquad A_{ij} = \frac{\partial_j \Lambda^i}{\Lambda^j - \Lambda^i}$$

which can be verified by a direct calculation. Geometric constructions underlying these formulae will be discussed below. The system (39) and (40) will be called the $\mathcal{L}^*_\alpha$-transform of system (18). Obviously, transformations $\mathcal{L}^*_\alpha$ preserve the semi-Hamiltonian property.

We also include the $\mathcal{L}^*_\alpha$ transforms of Lame coefficients h_i defined by the formulae

$$\partial_j \ln h_i = a_{ij} \qquad j \neq i.$$

The $\mathcal{L}^*_\alpha$-transformed Lame coefficients are given by

$$H_\alpha = h_\alpha \frac{\partial_\alpha \mu^\alpha}{\mu^\alpha}$$

$$H_i = h_i \left(1 - \frac{\mu^i}{\mu^\alpha}\right) \qquad i \neq \alpha.$$

One can check directly that

$$\partial_j \ln H_i = A_{ij} \qquad j \neq i.$$

Transformations $\mathcal{L}^*_\alpha$ and Laplace transformations $S_{\alpha\beta}$ satisfy the identities

$$\mathcal{L}^*_\alpha = \mathcal{L}^*_\beta \circ S_{\beta\alpha}.$$

To clarify the geometric picture underlying transformations $\mathcal{L}^*_\alpha$ we choose an arbitrary conservative representation

$$u^i_t = f^i_x$$

of system (18) and introduce the associated congruence

$$y^1 = u^1 y^0 - f^1$$
$$\vdots$$
$$y^n = u^n y^0 - f^n.$$

Let

$$u^i_\tau = q^i_x$$

be a commuting flow of system (18) with the characteristic velocities μ^i, so that

$$\partial_i q = \mu^i \partial_i u$$

(with the last identity holding for any $q = q^k, u = u^k$). Let us introduce the n-parameter family of 2-planes in A^{n+1} defined by the equations

$$\frac{y^1 - u^1 y^0 + f^1}{q^1} = \cdots = \frac{y^n - u^n y^0 + f^n}{q^n}. \tag{41}$$

The family of planes (41) possesses the following three important properties.

(a) Each plane π of the family (41) contains a line l of the initial congruence.

(b) Each plane π intersects the plane $\partial_i \pi$ along a line l_i:

$$l_i = \pi \cap \partial_i \pi$$

(we point out that two planes in A^{n+1} do not necessarily intersect along a line unless $n = 2$). Geometrically, this property implies that each one-parameter subfamily of (41) specified by fixing the values of $R^k, k \neq i$ envelopes a developable surface in A^{n+1}. The lines $l_i, i = 1, \ldots, n$, are called the characteristics of the plane π.

(c) Congruence l_i is conjugate to the ith focal hypersurface

$$\vec{r}_i = \left(\lambda^i, u^1\lambda^i - f^1, \ldots, u^n\lambda^i - f^n\right)$$

of the initial congruence l.

Conversely, one can show that any n-parameter family of 2-planes satisfying the properties (a)–(c) is necessarily of the form (41) for an appropriate commuting flow $u^i_\tau = q^i_x$. The congruence l_α will be called the $\mathcal{L}^*_\alpha$-transform of the initial congruence l. A direct calculation shows that l_α is representable in the form

$$y^1 = U^1 y^0 - F^1$$
$$\vdots$$
$$y^n = U^n y^0 - F^n$$

where

$$U^1 = u^1 - \frac{q^1}{\mu^\alpha} \qquad F^1 = f^1 - \frac{\lambda^\alpha q^1}{\mu^\alpha}$$
$$\vdots$$
$$U^n = u^n - \frac{q^n}{\mu^\alpha} \qquad F^n = f^n - \frac{\lambda^\alpha q^n}{\mu^\alpha}$$

(compare with theorem 4). The line l_α meets the focal hypersurface $\vec{r}_\alpha$ in the point

$$\left(\lambda^\alpha, u^1\lambda^\alpha - f^1, \ldots, u^n\lambda^\alpha - f^n\right).$$

The corresponding system of conservation laws

$$U^i_T = F^i_X$$

has the same Riemann invariants $R^1, \ldots, R^n$:

$$R^i_T = \Lambda^i R^i_X$$

(in fact, this is the analytic manifestation of the above property (c)), where the transformed characteristic velocities $\Lambda^i = \partial_i F/\partial_i U$ coincide with (40). Note that the final expressions for Λ^i do not depend on the particular conservative representation $u^i_t = f^i_x$ of system (18).

Obviously, the inverse transformation $l_\alpha \to l$ is the transformation $\mathcal{L}_\alpha$ of Lévy. Indeed, l_α is conjugate to the hypersurface $\vec{r}_\alpha$, while the initial congruence l consists of the R^α-tangents to the hypersurface $\vec{r}_\alpha$. Thus, Lévy transformations $\mathcal{L}_\alpha$ are the inverses of $\mathcal{L}^*_\alpha$. This can be demonstrated analytically as well.

Let us consider a system

$$R^i_t = \lambda^i R^i_x$$

along with its Lévy transform $\mathcal{L}_\alpha$ defined by the formulae (33) and (34). The transformed system (33) and (34) possesses the commuting flow

$$\mu^\alpha = \frac{1}{h}$$
$$\mu^i = \frac{a_{i\alpha}}{a_{i\alpha}h - \partial_\alpha h} \qquad i \neq \alpha$$

(which can be obtained by a shift $g \to g+1$ in the formulae (34)). Applying to the transformed system (33) and (34) transformation $\mathcal{L}^*_\alpha$ (generated by the above commuting flow), we return to the initial system

$$R^i_t = \lambda^i R^i_x.$$

Conversely, let us consider transformation $\mathcal{L}^*_\alpha$. The transformed system (39) and (40) possesses the conservation law

$$h_T = g_X \qquad h = \frac{1}{\mu^\alpha} \qquad g = \frac{\lambda^\alpha}{\mu^\alpha}$$

(which can be obtained by a shift $q \to q - 1$ in the formula (38)). Applying to (39) and (40) the transformation $\mathcal{L}_\alpha$ (generated by this particular h), we also return back to the initial system.

Acknowledgments

I would like to thank M V Pavlov for useful discussions. This research was partially supported by the INTAS 96-0770 and the RFFI grant no 99-01-00010.

Appendix. Ribaucour congruences of spheres

Let M^n be a hypersurface in the Euclidean space E^{n+1} parametrized by coordinates $u^1, \ldots, u^n$. Let $\vec{r}$ and $\vec{n}$ be the radius-vector and the unit normal of M^n, respectively. The Weingarten formulae

$$\frac{\partial \vec{n}}{\partial u^j} = w^i_j(u) \frac{\partial \vec{r}}{\partial u^i}$$

define the so-called Weingarten (shape) operator of hypersurface M^n. Its eigenvalues and eigenvectors are called the principal curvatures and the principal directions of M^n, respectively. Let us consider an n-parameter family (congruence) of hyperspheres S which are tangent to M^n. Let $R(u)$ be the radius of the hypersphere from the congruence S which is tangent to M^n at the point $\vec{r}(u)$. The congruence of hyperspheres S has exactly two enveloping hypersurfaces (note the difference with line congruences!) one of which coincides with M^n by a construction. Let $\tilde{M}^n$ be the second sheet of the envelope. Clearly, the congruence S induces the point correspondence between both sheets: a point $p \in M^n$ is said to correspond to the point $\tilde{p} \in \tilde{M}^n$ if p and $\tilde{p}$ are the two points of tangency of the envelopes with one and the same hypersphere from the congruence S.

Definition. A congruence of hyperspheres S is called the congruence of Ribaucour if the principal distributions of M^n correspond to the principal distributions of $\tilde{M}^n$.

Let us introduce the system of hydrodynamic type

$$u^i_t = w^i_j(u) u^j_x \tag{A1}$$

where w^i_j is the Weingarten operator of M^n. We refer to [8] for a general discussion of the correspondence between hypersurfaces and Hamiltonian systems of hydrodynamic type. Let

$$h(u)_t = g(u)_x$$

be a conservation law of system (A1).

Theorem 4. *A congruence $S(u)$ is the congruence of Ribaucour if and only if $R(u)$ is representable in the form*

$$R(u) = \frac{h(u)}{g(u)}$$

for some conservation law of the system (A1).

In the case $n = 2$ this result (stated in a somewhat different form) can be found in [6]. It should be emphasized that this theorem applies equally to hypersurfaces which do not possess a curvature-line parametrization (for $n = 2$ such parametrization is always possible). We hope to present the details elsewhere.

References

[1] Agafonov S I and Ferapontov E V 1996 *Izv. Ross. Akad. Nauk, Ser. Mat.* **60** 3–30
[2] Agafonov S I and Ferapontov E V 1999 *J. Math. Sci.* **94** 1748–92
[3] Akivis M A and Goldberg V V 1993 *Projective Differential Geometry of Submanifolds (Mathematics Library) vol 49* (Amsterdam: North-Holland)
[4] Doliva A, Mañas M, Alonso L M, Medina E and Santini P M 1998 Charged free fermions, vertex operators and classical theory of conjugate nets *Preprint* solv-int/9803015
[5] Dubrovin B A and Novikov S P 1989 *Usp. Mat. Nauk* **44** 29–98
[6] Eisenhart L P 1962 *Transformations of Surfaces* 2nd edn (New York: Chelsea)
[7] Ferapontov E V 1997 *J. Phys. A: Math. Gen.* **30** 6861–78
[8] Ferapontov E V 1995 *Diff. Geom. Appl.* **5** 121–52
[9] Finikov S P 1950 *Theory of Congruences* (M-L Gostekhizdat)
[10] Forsyth A R 1906 *Theory of Differential Equations* (Cambridge: Cambridge University Press) p 6
[11] Kamran N and Tenenblat K 1996 *Duke Math. J.* **84** 237–66
[12] Lévy M L 1886 Quelques équations linéaires aux derivées partielles du second *J. École Polytech.* **56** 63–7
[13] Sévennec B 1994 Géométrie des systèmes hyperboliques de lois de conservation *Mémoire (Nouvelle série) Bull. Soc. Math. Fr. Suppl.* **122** 1–125
[14] Tsarev S P 1991 *Math. USSR Izv.* **37** 397–419
[15] Temple B 1983 *Trans. Am. Math. Soc.* **280** 781–95